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LATERAL BUCKLING OF A CANTILEVER SUBJECTED TO A TRANSVERSE FOLLOWER FORCE

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Abstract—In this note we study the stability of bending-torsional equilibrium of a cantilevered bar subjected at its end section to a follower force; this case may represent a wing subjected to the jet of a turbine-engine.

Making use of the dynamic analysis, we can determine the critical thrust.

1. INTRODUCTION

THE PROBLEM of stability of the elastic equilibrium of systems subject to non conservative forces is becoming more and more important in the theory of structures, because of the rapid and continuous progress of modern technology. It is thus necessary to consider still more types of forces acting on the structure, and the traditional loads occurring in the classical theory of stability of equilibrium, that is forces having a potential and usually caused by the dead load, can be considered only as a particular type among all forces studied in modern engineering, especially in the fields of mechanics, aeronautics and rocketry.

Such developments brought a revision of certain principles governing the equilibrium stability, which were acquired in the technical practice because of the current use of research methods the validity of which, for certain fields, is usually taken for granted. Some discrepancies were discovered concerning the equilibrium stability of the axes of propellers and turbines subject to torsion and, consequently, the soundness of the static approach was discussed. As a result, Ziegler's theory [1] appeared to be a basic one; his subtle analysis pointed out the various aspects of the problem with reference to the more general definitions of the equilibrium stability, in accordance with Poincaré and Liapunov, in connection with the study of the motion characteristics of a perturbed system.

To the static concept of Euler's instability, caused in an elastic system by conservative forces, which contains the physical phenomenon by which the system loses its elastic reactivity because of the destabilizing effects of applied forces, a new concept was added, considering the possibility of "resonance" conditions between the nonconservative forces, due to the deformation of the system, and the system itself subject to oscillation.

Various problems have been studied, starting from the basic case (one of the first ones analysed) of a cantilever subjected to an axial "follower" force, that is a force acting always along the tangent to the elastic line at the end section. Some of these problems are: in the field of mechanics, the equilibrium stability of shafts subjected to torsion with torque having a vector axis constant or variable [2–4] and, in the field of aeronautics, the stability of plates, cylinders, etc. oscillating in gas flow [5].

In our opinion, among all cases of nonconservative forces encountered in technology particularly interesting is the "follower" force, following the deformation of the system to which it is applied. Such force idealizes the impulse of a jet-engine applied to the structure. Therefore we want to study, in this note, a stability problem which, to our knowledge, has not been studied as yet; that is the bending-torsional equilibrium stability of a cantilever subjected to a transverse "follower" force applied at its end section.

Generalizing from the conservative to the nonconservative field, a typical stability problem studied by Prandtl and Michell, we examine a case of instability which describes the behaviour of a wing supporting a jet-engine. Such study could be a starting point for some researches in the field of stability of wings supporting jet-engines when jet effects are combined with aerodynamic effects.

2. DIFFERENTIAL EQUATIONS GOVERNING THE PROBLEM

Let us consider a cantilever having a thin rectangular section and length l as shown in Fig. 1.

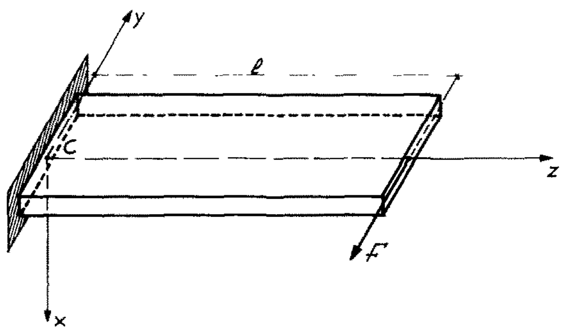

FIG. 1. Cantilever subjected to a follower force at its end section.

With reference to the fixed Cartesian coordinate system of the figure $Cxyz$, C and EI_y are the torsional and the flexural rigidity around the z and y axes, respectively, and $\mathbf{F}$ is the "follower" force applied at the end $z = l$.

We consider $\mathbf{F}$ following the η axis which represents the y axis moving with the cross sections of the beam, as shown in Fig. 2. Furthermore, we consider

$$u = u(z, t) \tag{1}$$

to be the displacement component along x, which is positive if in the positive direction of x, of the beam axis describing the flexural oscillation around y, and

$$\varphi = \varphi(z, t) \tag{2}$$

to be the component describing the torsional oscillation, which is positive in the case of Fig. 2.

For $z = l$, (1) and (2) are u_1 and φ_1.

Considering the equilibrium of the portion of the cantilever to the right of section z in the deformed condition, characterized by a deflection around y, (1), and a torsion (2), the moments with respect to fixed axes x, y, and z of force $\mathbf{F}$, now inclined by φ_1 with respect to y (Fig. 2), are given by:

$$\begin{aligned} M_x &= F(l-z)\cos\varphi_1 \\ M_y &= -F(l-z)\sin\varphi_1 \\ M_z &= -F(u_1-u)\cos\varphi_1 \end{aligned} \tag{3}$$

that is, neglecting small quantities of higher order than the first,

$$\begin{aligned} M_x &= -F(l-z) \\ M_y &= -F(l-z)\varphi_1 \\ M_z &= -F(u_1-u). \end{aligned} \tag{4}$$

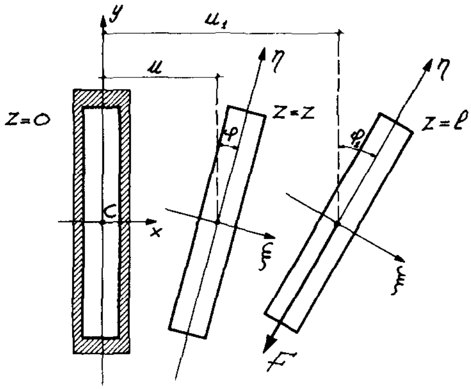

FIG. 2. Position of the follower force with respect to the cantilever cross sections.

Calculating the components of moments (4) around axes ξ, η, ζ, where ξ and η are the main inertia axes of displaced sections, and ζ is the tangent to the deformed beam axis during the oscillation, we have, neglecting small quantities of higher order than the first,

$$\begin{aligned} M_\xi &= -F(l-z) \\ M_\eta &= F(l-z)(\varphi_1-\varphi) \\ M_\zeta &= -F\frac{\partial u}{\partial z}(l-z)+F(u_1-u). \end{aligned} \tag{5}$$

We must observe that in (5), unlike the typical conservative case of Prandtl and Michell, in M_η there appears the term $F(l-z)\varphi_1$ due to rotation φ_1 of **F**.

In order to complete the differential equations of the problem, we must now evaluate the inertia forces.

If we think of the main application of the problem in technology, we cannot omit considering the presence of a mass m at the end section $z = l$ (Fig. 3).

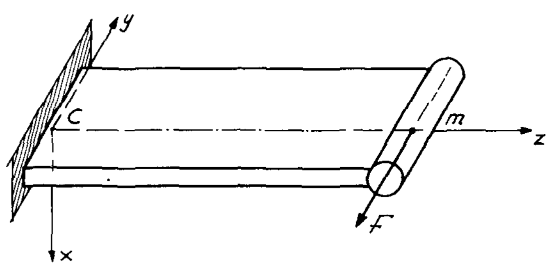

FIG. 3. Distribution of masses for a cantilever.

Particularly in the case we are studying, that is the wing stability under the jet of a jet-engine applied at the end section, we will not make a considerable error, trying to find $\mathbf{F}_{crit}$, and we will simplify considerably the analytic procedure, by neglecting the effects of the mass distributed by the cantilever with respect to the concentrated mass m which we consider having a mass inertia moment I_z.

With such simplification and with (5) we will have the following equations which govern the bending-torsional dynamics of the cantilever:

$$
\begin{aligned}
&EI_y\frac{\partial^2 u}{\partial z^2}-F(l-z)(\varphi_1-\varphi)-m(l-z)\frac{\partial^2 u_1}{\partial t^2}=0\\
&-C\frac{\partial\varphi}{\partial z}+F(l-z)\frac{\partial u}{\partial z}-F(u_1-u)+I_z\frac{\partial^2\varphi_1}{\partial t^2}=0.
\end{aligned}
\tag{6}
$$

3. INTEGRATION OF THE DIFFERENTIAL EQUATIONS SYSTEM (6)

With the assumption

$$
\begin{aligned}
u(z,t)&=U(z)\exp i\omega t \qquad & \varphi(z,t)&=\Phi(z)\exp i\omega t\\
u_1(t)&=U_1\exp i\omega t & \varphi_1(t)&=\Phi_2\exp i\omega t
\end{aligned}
\tag{7}
$$

we obtain from equations (6) a system of ordinary differential equations

$$
\begin{aligned}
&EI_y\frac{\mathrm{d}^2 U}{\mathrm{d}z^2}-F(l-z)(\Phi_1-\Phi)+m(l-z)\omega^2 U_1=0\\
&-C\frac{\mathrm{d}\Phi}{\mathrm{d}z}+F(l-z)\frac{\mathrm{d}U}{\mathrm{d}z}-F(U_1-U)-I_z\omega^2\Phi_1=0.
\end{aligned}
\tag{8}
$$

Solving for $\frac{\mathrm{d}^2U}{\mathrm{d}z^2}$ from the first equation (8), we have:

$$
\frac{\mathrm{d}^2U}{\mathrm{d}z^2}=\frac{F}{EI_y}(l-z)(\Phi_1-\Phi)-\frac{m}{EI_y}(l-z)\omega^2U_1. \tag{9}
$$

Then, differentiating the second equation (8) and substituting into it equation (9), we obtain:

$$
-C\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2}+\frac{F^2}{EI_y}(l-z)^2(\Phi_1-\Phi)-\frac{F}{EI_y}(l-z)^2 m\omega^2 U_1=0 \tag{10}
$$

the resulting equation for Φ (z).

Setting

$$
s=l-z \tag{11}
$$

and

$$
\frac{F^2}{CEI_y}=\nu^4 \tag{12}
$$

equation (1) is changed into:

$$\frac{d^2\Phi}{ds^2}+\nu^4 s^2\Phi = -\nu^4 s^2\left(\frac{m\omega^2}{F}U_1-\Phi_1\right) \tag{13}$$

which admits the general integral

$$\Phi(s) = A_0\alpha_0(s)+A_1\alpha_1(s)-\left(\frac{m\omega^2}{F}U_1-\Phi_1\right) \tag{14}$$

where $\alpha_0(s)$ and $\alpha_1(s)$ are represented by the infinite series:

$$\alpha_0(s) = 1-\frac{\nu^4 s^4}{3.4}+\frac{\nu^8 s^8}{3.4.7.8}-\frac{\nu^{12}s^{12}}{3.4.7.8.11.12}+\dots \tag{15}$$

$$\alpha_1(s) = s-\frac{\nu^4 s^5}{4.5}+\frac{\nu^8 s^9}{4.5.8.9}-\frac{\nu^{12}s^{13}}{4.5.8.9.12.13}+\dots \tag{16}$$

The constants A_0 and A_1 of equation (14) will be determined by applying the boundary conditions

$$C\left(\frac{d\Phi}{ds}\right)_{s=0} = I_z\omega^2\Phi_1 \tag{17}$$

and

$$(\Phi)_{s=l} = 0. \tag{18}$$

We obtain for $\Phi(s)$ the final expression:

$$\Phi(s) =\left\{\Phi_1\left[-\frac{I_z\omega^2}{C}\frac{\alpha_1(l)}{\alpha_0(l)}-\frac{1}{\alpha_0(l)}\right]+U_1\frac{m\omega^2}{F\alpha_0(l)}\right\}\alpha_0(s)+\frac{I_z\omega^2}{C}\Phi_1\alpha_1(s)-\frac{1}{F}(m\omega^2 U_1-F\Phi_1). \tag{19}$$

By substituting into equation (9) equation (19) with twice repeated integration, we obtain the expression for $U(s)$

$$U(s) = \frac{F\Phi_1}{EI_y}\frac{s^3}{2.3}-\frac{F}{EI_y}J_2(s)-\frac{m\omega^2}{EI_y}U_1\frac{s^3}{2.3}+A_2 s+A_3 \tag{20}$$

where the two constants A_2 and A_3 will be determined by applying the boundary conditions to $U(s)$.

The function $J_2(s)$, which appears in equation (20), derives from the repeated integration of equation (19). Using the notation:

$$\begin{aligned} J_1(s) &= \int s\Phi(s)\,ds \\ &= \left\{\left[-\frac{I_z\omega^2}{C}\frac{\alpha_1(l)}{\alpha_0(l)}-\frac{1}{\alpha_0(l)}\right]\Phi_1+\frac{m\omega^2}{F\alpha_0(l)}U_1\right\}\left(\frac{s^2}{2}-\frac{\nu^4}{3.4}\frac{s^6}{6}+\frac{\nu^8}{3.4.7.8}\frac{s^{10}}{10}\right. \\ &\left.-\dots\right)+\frac{I_z\omega^2}{C}\Phi_1\left(\frac{s^3}{3}-\frac{\nu^4}{4.5}\frac{s^7}{7}+\frac{\nu^8}{4.5.8.9}\frac{s^{11}}{11}-\dots\right)-\frac{1}{F}(m\omega^2 U_1-F\Phi_1)\frac{s^2}{2} \end{aligned} \tag{21}$$

we have:

$$J_2(s) = \int J_1(s)\,ds$$

$$= \left\{\left[-\frac{I_z\omega^2}{C}\frac{\alpha_1(l)}{\alpha_0(l)} - \frac{1}{\alpha_0(l)}\right]\Phi_1 + \frac{m\omega^2}{F\alpha_0(l)}U_1\right\}\left(\frac{1}{2}\frac{s^3}{3} - \frac{v^4}{3.4.6}\frac{s^7}{7} + \frac{v^8}{3.4.7.8.10}\frac{s^{11}}{11} - \cdots\right)$$
$$+\frac{I_z\omega^2}{C}\Phi_1\left(\frac{1}{3}\frac{s^4}{4} - \frac{v^4}{4.5.7}\frac{s^8}{8} + \frac{v^8}{4.5.8.9.11}\frac{s^{12}}{12} - \cdots\right) - \frac{1}{F}(m\omega^2U_1 - F\Phi_1)\frac{s^3}{2.3}. \tag{22}$$

The boundary conditions:

$$(U)_{s=l} = 0 \tag{23}$$

$$\left(\frac{dU}{ds}\right)_{s=l} = 0 \tag{24}$$

finally give:

$$U(s) = \frac{F\Phi_1}{EI_y}\frac{s^3}{2.3} - \frac{F}{EI_y}J_2(s) - \frac{m\omega^2}{EI_y}U_1\frac{s^3}{2.3} + \left[\frac{F}{EI_y}J_1(l) - \frac{l^2}{2EI_y}(F\Phi_1 - m\omega^2U_1)\right]s$$
$$+\left[\frac{l^3}{3EI_y}(F\Phi_1 - m\omega^2U_1) - \frac{F}{EI_y}[lJ_1(l) - J_2(l)]\right]. \tag{25}$$

In this manner we obtain the functions $\Phi = \Phi(s)$ and $U = U(s)$ which integrate the differential equations system (6) and satisfy the boundary conditions (17), (18), (23), and (24).

4. DETERMINATION OF CRITICAL FORCE

With equations (19) and (25) we have now the equations of bending-torsional oscillation "frequencies" of the cantilever by applying the conditions:

$$(U)_{s=0} = U_1 \tag{26}$$

$$(\Phi)_{s=0} = \Phi_1 \tag{27}$$

which are explicitly represented by the system:

$$\frac{l^3}{3EI_y}(F\Phi_1 - m\omega^2U_1) - \frac{F}{EI_y}[lJ_1(l) - J_2(l)] = U_1$$
$$-\Phi_1\left[\frac{I_z\omega^2}{C}\frac{\alpha_1(l)}{\alpha_0(l)} + \frac{1}{\alpha_0(l)}\right] + U_1\frac{m\omega^2}{F\alpha_0(l)} - \frac{1}{F}(m\omega^2U_1 - F\Phi_1) = \Phi_1 \tag{28}$$

which is linear and homogeneous in the unknowns U_1 and Φ_1.

To simplify, if we denote:

$$\rho_1 = \frac{1}{2} - \frac{\psi}{3.4.6} + \frac{\psi^2}{3.4.7.8.10} - \cdots$$

$$\rho_2 = \frac{1}{3} - \frac{\psi}{3.4.6.7} + \frac{\psi^2}{3.4.7.8.10.11} - \cdots$$

$$\rho_3 = \frac{1}{2.3} - \frac{\psi}{4.5.7.8} + \frac{\psi^2}{4.5.8.9.11.12} - \cdots \tag{29}$$

$$\rho_4 = \frac{1}{3.4} - \frac{\psi}{4.5.7.8} + \frac{\psi^2}{4.5.8.9.11.12} - \cdots$$

where ψ is the nondimensional quantity

$$\psi = v^4 l^4 = \frac{F^2 l^4}{CEI_y} \tag{30}$$

we can write the system (28) as follows:

$$\begin{gathered} U_1\left[1+\frac{\Omega^2}{\alpha_0(l)}(\rho_1-\rho_3)\right]+\Phi_1\frac{Fl^3}{EI_y}\left\{-\left[\frac{I_z\omega^2}{C}\frac{\alpha_1(l)}{\alpha_0(l)}+\frac{1}{\alpha_0(l)}\right](\rho_1-\rho_3)+\frac{I_z\omega^2}{C}l(\rho_2-\rho_4)\right\}=0 \\ U_1\frac{m\omega^2}{F}\left(\frac{1}{\alpha_0(l)}-1\right)+\Phi_1\left\{-\frac{1}{\alpha_0(l)}\left[\frac{I_z\omega^2}{C}\alpha_1(l)+1\right]\right\}=0 \end{gathered} \tag{31}$$

where we denote the frequency ω by the nondimensional quantity:

$$\Omega^2 = \frac{m\omega^2 l^3}{EI_y}. \tag{32}$$

As a result we have the condition:

$$\begin{vmatrix} \left[1+\Omega^2\frac{\rho_1-\rho_3}{\alpha_0(l)}\right] & \frac{Fl^3}{EI_y}\left\{-\left[\frac{I_z\omega^2}{C}\frac{\alpha_1(l)}{\alpha_0(l)}+\frac{1}{\alpha_0(l)}\right](\rho_1-\rho_3)+\frac{I_z\omega^2}{C}l(\rho_2-\rho_4)\right\} \\ \frac{m\omega^2}{F}\left[\frac{1}{\alpha_0(l)}-1\right] & -\frac{1}{\alpha_0(l)}\left[\frac{I_z\omega^2}{C}\alpha_1(l)+1\right] \end{vmatrix} = 0 \tag{33}$$

and, expanding the determinant, we obtain:

$$\Omega^4\{\tilde{\alpha}_1(l)(\rho_3-\rho_1)+[1-\alpha_0(l)](\rho_4-\rho_2)\}\vartheta+\Omega^2[-\tilde{\alpha}_1(l)\vartheta+(\rho_3-\rho_1)]-1=0 \tag{34}$$

where we used the notations:

$$\tilde{\alpha}_1(l) = \frac{\alpha_1(l)}{l} \tag{35}$$

and

$$\vartheta = \frac{I_z}{ml^2}\frac{EI_y}{C}. \tag{36}$$

Equation (34) for the various values of quantities ψ and ϑ, which indicate the applied force **F**, and the geometrical and mechanical characteristics of the cantilever (wing), gives the frequencies ω of bending torsional oscillation.

First of all, before determining the possible $\mathbf{F}_{\text{crit}}$, we can observe that, with a wrong formulation of the nonconservative problem, by applying the static method, no doubt the system would appear always stable. In fact, if we have $\omega = 0$, the determinant of (33) could never equal zero, thus we would have only $U(s) = \Phi(s) = 0$.

On the contrary, the critical condition is obtained by studying the dynamic problem; in fact, if Δ is the discriminant of equation (34), the critical value of **F** is the value which annuls Δ.

The condition:

$$\Delta = 0 \tag{37}$$

can be expressed, with equation (34), as follows:

$$[\tilde{\alpha}_1(l)\vartheta + (\rho_3 - \rho_1)]^2 = 4[\alpha_0(l) - 1]\vartheta(\rho_4 - \rho_2). \tag{38}$$

Equation (38) for the unknown ψ given by equation (30), is difficult to solve because of the type of equations (29).

We can simplify it considerably by observing that quantity ϑ is very small compared with unity. In fact, it is approximately $\frac{1}{1000}$ or even less.

Therefore, in order to determine $\mathbf{F}_{\mathrm{crit}}$, it will not be a great mistake to replace equation (38) by the simpler condition:

$$\rho_3 - \rho_1 = 0 \tag{39}$$

which, with equations (29) and (39), can be expressed as follows:

$$\begin{aligned}\rho_3 - \rho_1 = -\frac{1}{3}\Bigg(1 - \frac{\psi}{4.7} + \frac{\psi^2}{4.7.8.11} - \frac{\psi^3}{4.7.8.11.12.15} \\ + \frac{\psi^4}{4.7.8.11.12.15.16.19} - \cdots\Bigg) = 0.\end{aligned} \tag{40}$$

It is found that root ψ_{crit} lies between 47 and 49.

A more accurate calculation, supported by equation (38), with acceptable approximation for $\vartheta \leqslant 10^{-3}$, gives:

$$\psi_{\mathrm{crit}} \equiv 48{\cdot}80. \tag{41}$$

Therefore we have the critical force:

$$\boxed{F_{\mathrm{crit}} = \frac{6{\cdot}99}{l^2}\sqrt{(EI_y C)}} \tag{42}$$

The changes of the characteristic exponent of equation (7) with force **F** is shown in Fig. 4, which represents the complex plane.

If **F** = 0, we have the four values of the exponent:

$$i\omega_1; \qquad i\omega_2; \qquad -i\omega_1; \qquad -i\omega_2$$

corresponding to the separated flexural and torsional oscillations.

On the other hand, the presence of **F** in equation (34) couples the flexural and torsional oscillations and $\mathbf{F}_{\mathrm{crit}}$, given by equation (47), leads to the double root

$$\pm i\omega_{\mathrm{crit}}$$

represented in the figure by the joining point of the two arrows. For $\mathbf{F} > \mathbf{F}_{\mathrm{crit}}$ the characteristic exponents leave the imaginary axis and become complex. In this case the presence of the characteristic exponent having a positive real part indicates loss of stability by oscillations with increasing amplitude.

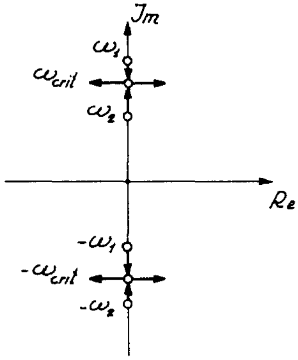

FIG. 4. Characteristic exponents in the complex plane.

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Résumé—Dans cette note nous étudions la stabilité de l'équilibre de flexion-torsion d'une barre en console, soumise à sa extremité à une force suivante; ce cas peut representer une aile soumise a la poussée d'un moteur à fusée.

En faisant usage de l'analyse dynamique, nous pouvons determiner la poussée critique.

Zusammenfassung—In dieser Anmerkung studieren wir die Stabilität des Biegungs-Verdrehungs Gleichgewichts eines Kragträgers, welche an freiem Ende einer mitgehenden Kraft unterworfen ist; dieser Fall kann einen Flügel darstellen, welcher dem Schub eines Turbinenmotors ausgesetzt ist.

Bei Verwendung der dynamischen Analyse, können wir den kritischen Schub bestimmen.

Абстракт— В зтой записке изучается устойчивость изгибающе-крутильного равновесия консольного стержня, подвергнутого в его конечной секции ведомой следчщей силе; зтот случай может представлять из себя крыло, подвергнутое действию турбо-реактивного двигателя. Применяя динамический анализ, мы можем определить критическую силу тяги.